CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

1. Calculus is the mathematics of change. Precalculus is more static. Answers will vary. *Sample answer:*

Precalculus: Area of a rectangle
Calculus: Area under a curve

Precalculus: Work done by a constant force
Calculus: Work done by a variable force

Precalculus: Center of a rectangle
Calculus: Centroid of a region

2. A secant line through a point P is a line joining P and another point Q on the graph.

The slope of the tangent line P is the limit of the slopes of the secant lines joining P and Q, as Q approaches P.

3. Precalculus: $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

4. Calculus required: Velocity is not constant.

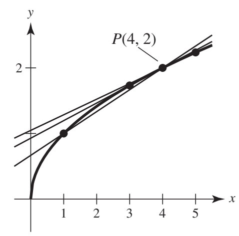
Distance $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

5. Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

6. Precalculus: rate of change = slope = 0.08

7. $f(x) = \sqrt{x}$

(a)



(b) $$\text{slope} = m = \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)} = \frac{1}{\sqrt{x} + 2}, \ x \neq 4$$

$$x = 1: m = \frac{1}{\sqrt{1} + 2} = \frac{1}{3}$$

$$x = 3: m = \frac{1}{\sqrt{3} + 2} \approx 0.2679$$

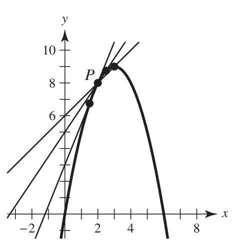
$$x = 5: m = \frac{1}{\sqrt{5} + 2} \approx 0.2361$$

(c) At $P(4, 2)$ the slope is $\frac{1}{\sqrt{4} + 2} = \frac{1}{4} = 0.25$.

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

8. $f(x) = 6x - x^2$

(a)



(b) $$\text{slope} = m = \frac{(6x - x^2) - 8}{x - 2} = \frac{(x - 2)(4 - x)}{x - 2} = (4 - x), \ x \neq 2$$

For $x = 3, m = 4 - 3 = 1$

For $x = 2.5, m = 4 - 2.5 = 1.5 = \frac{3}{2}$

For $x = 1.5, m = 4 - 1.5 = 2.5 = \frac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of x close to 2.

9. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. Answers will vary. *Sample answer*:

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

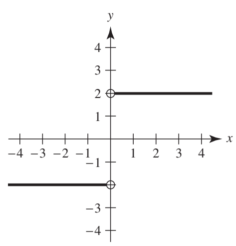
$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

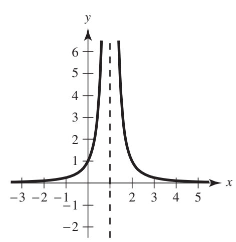
Section 1.2 Finding Limits Graphically and Numerically

1. As the graph of the function approaches 8 on the horizontal axis, the graph approaches 25 on the vertical axis.

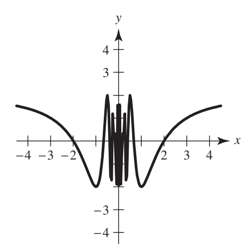
2. (i) The values of f approach different numbers as x approaches c from different sides of c:



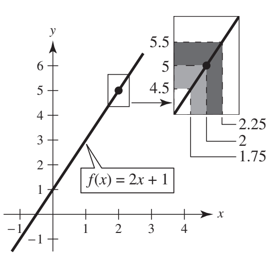
(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



3.



4. No. For example, consider Example 2 from this section.

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim_{x\to 2} f(x) = 1$, but $f(2) = 0$

5.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	0.3448	0.3344	0.3334	?	0.3332	0.3322	0.3226

$\lim_{x\to 4} \frac{x-4}{x^2-5x-4} \approx 0.3333$ $\left(\text{Actual limit is } \frac{1}{3}.\right)$

6.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	0.1695	0.1669	0.1667	?	0.1666	0.1664	0.1639

$\lim_{x\to 3} \frac{x-3}{x^2-9} \approx 0.1667$ $\left(\text{Actual limit is } \frac{1}{6}.\right)$

7.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

8.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	–0.0641	–0.0627	–0.0625	–0.0625	–0.0623	–0.0610

$\lim_{x\to 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625$ $\left(\text{Actual limit is } -\frac{1}{16}.\right)$

9.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

10.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	–0.0005	–0.0050	–0.0500

$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

11.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

12.

x	–4.1	–4.01	–4.001	–4	–3.999	–3.99	–3.9
$f(x)$	1.1111	1.0101	1.0010	?	0.9990	0.9901	0.9091

$\lim_{x\to -4} \frac{x+4}{x^2+9x+20} \approx 1.0000$ (Actual limit is 1.)

13.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666$ $\left(\text{Actual limit is } \frac{2}{3}.\right)$

14.

x	–3.1	–3.01	–3.001	–3	–2.999	–2.99	–2.9
$f(x)$	27.91	27.0901	27.0090	?	26.9910	26.9101	26.11

$\lim_{x\to -3} \frac{x^3+27}{x+3} \approx 27.0000$ (Actual limit is 27.)

15.

x	–6.1	–6.01	–6.001	–6	–5.999	–5.99	–5.9
$f(x)$	–0.1248	–0.1250	–0.1250	?	–0.1250	–0.1250	–0.1252

$\lim_{x\to -6} \dfrac{\sqrt{10-x}-4}{x+6} \approx -0.1250$ $\left(\text{Actual limit is } -\dfrac{1}{8}.\right)$

16.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$\lim_{x\to 2} \dfrac{x/(x+1)-2/3}{x-2} \approx 0.1111$ $\left(\text{Actual limit is } \dfrac{1}{9}.\right)$

17.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$\lim_{x\to 0} \dfrac{\sin 2x}{x} \approx 2.0000$ (Actual limit is 2.) (Make sure you use radian mode.)

18.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$\lim_{x\to 0} \dfrac{\tan x}{\tan 2x} \approx 0.5000$ $\left(\text{Actual limit is } \dfrac{1}{2}.\right)$

19. $f(x) = \dfrac{2}{x^3}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–2000	-2×10^6	-2×10^9	?	2×10^9	2×10^6	2000

As x approaches 0 from the left, the function decreases without bound. As x approaches 0 from the right, the function increases without bound.

20. $f(x) = \dfrac{3|x|}{x^2}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	30	300	3000	?	3000	300	30

As x approaches 0 from either side, the function increases without bound.

21. $\lim_{x\to 3}(4-x) = 1$

22. $\lim_{x\to 0} \sec x = 1$

23. $\lim_{x\to 2} f(x) = \lim_{x\to 2}(4-x) = 2$

24. $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2+3) = 4$

25. $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$ does not exist.

For values of x to the left of 2, $\dfrac{|x-2|}{(x-2)} = -1$, whereas

for values of x to the right of 2, $\dfrac{|x-2|}{(x-2)} = 1$.

26. $\lim_{x\to 5} \dfrac{2}{x-5}$ does not exist because the function increases and decreases without bound as x approaches 5.

27. $\lim_{x \to 0} \cos(1/x)$ does not exist because the function oscillates between –1 and 1 as x approaches 0.

28. $\lim_{x \to \pi/2} \tan x$ does not exist because the function increases without bound as x approaches $\frac{\pi}{2}$ from the left and decreases without bound as x approaches $\frac{\pi}{2}$ from the right.

29. (a) $f(1)$ exists. The black dot at (1, 2) indicates that $f(1) = 2$.

(b) $\lim_{x \to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x \to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x \to 4} f(x) = 2$.

30. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at –2.

(b) $\lim_{x \to -2} f(x)$ does not exist. As x approaches –2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x \to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

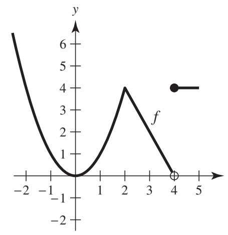
(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x \to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x \to 2} f(x) = \frac{1}{2}$.

(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

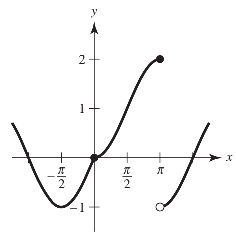
(h) $\lim_{x \to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

31.



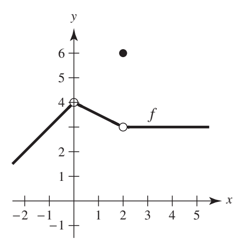
$\lim_{x \to c} f(x)$ exists for all values of $c \neq 4$.

32.

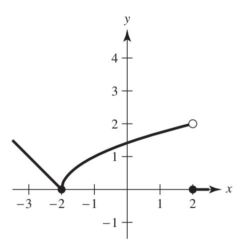


$\lim_{x \to c} f(x)$ exists for all values of $c \neq \pi$.

33. One possible answer is



34. One possible answer is



35. You need $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$, as desired.

36. You need $\left|f(x) - 1\right| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < 0.01$. Let $\delta = \frac{1}{101}$. If $0 < \left|x - 2\right| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow \left|x - 1\right| > \frac{100}{101}$$

and you have

$$\left|f(x) - 1\right| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

37. You need to find δ such that $0 < \left|x - 1\right| < \delta$ implies

$\left|f(x) - 1\right| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < \left|x - 1\right| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$\left|f(x) - 1\right| = \left|\frac{1}{x} - 1\right| < 0.1.$$

38. You need to find δ such that $0 < \left|x - 1\right| < \delta$ implies

$\left|f(x) - 1\right| = \left|2 - \frac{1}{x} - 1\right| = \left|1 - \frac{1}{x}\right| < \varepsilon.$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$
$$1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon$$
$$\frac{1}{1-\varepsilon} > x > \frac{1}{1+\varepsilon}$$
$$\frac{1}{1-\varepsilon} - 1 > x - 1 > \frac{1}{1+\varepsilon} - 1$$
$$\frac{\varepsilon}{1-\varepsilon} > x - 1 > \frac{-\varepsilon}{1+\varepsilon}$$

For $\varepsilon = 0.05$, take $\delta = \frac{0.05}{1 - 0.05} \approx 0.05$.

For $\varepsilon = 0.01$, take $\delta = \frac{0.01}{1 - 0.01} \approx 0.01$.

For $\varepsilon = 0.005$, take $\delta = \frac{0.005}{1 - 0.005} \approx 0.005$.

As ε decreases, so does ε.

39. $\lim_{x\to 2}(3x+2) = 3(2)+2 = 8 = L$

(a) $\left|(3x+2)-8\right| < 0.01$

$\left|3x-6\right| < 0.01$

$3\left|x-2\right| < 0.01$

$0 < \left|x-2\right| < \frac{0.01}{3} \approx 0.0033 = \delta$

So, if $0 < \left|x-2\right| < \delta = \frac{0.01}{3}$, you have

$3\left|x-2\right| < 0.01$

$\left|3x-6\right| < 0.01$

$\left|(3x+2)-8\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|(3x+2)-8\right| < 0.005$

$\left|3x-6\right| < 0.005$

$3\left|x-2\right| < 0.005$

$0 < \left|x-2\right| < \dfrac{0.005}{3} \approx 0.00167 = \delta$

Finally, as in part (a), if $0 < \left|x-2\right| < \dfrac{0.005}{3}$, you have $\left|(3x+2)-8\right| < 0.005$.

40. $\lim_{x\to 6}\left(6-\dfrac{x}{3}\right) = 6-\dfrac{6}{3} = 4 = L$

(a) $\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.01$

$\left|2-\dfrac{x}{3}\right| < 0.01$

$\left|-\frac{1}{3}(x-6)\right| < 0.01$

$\left|x-6\right| < 0.03$

$0 < \left|x-6\right| < 0.03 = \delta$

So, if $0 < \left|x-6\right| < \delta = 0.03$, you have

$\left|-\frac{1}{3}(x-6)\right| < 0.01$

$\left|2-\dfrac{x}{3}\right| < 0.01$

$\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.005$

$\left|2-\dfrac{x}{3}\right| < 0.005$

$\left|-\dfrac{1}{3}(x-6)\right| < 0.005$

$\left|x-6\right| < 0.015$

$0 < \left|x-6\right| < 0.015 = \delta$

As in part (a), if $0 < \left|x-6\right| < 0.015$, you have

$\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.005.$

41. $\lim_{x\to 2}\left(x^2-3\right) = 2^2-3 = 1 = L$

(a) $\left|\left(x^2-3\right)-1\right| < 0.01$

$\left|x^2-4\right| < 0.01$

$\left|(x+2)(x-2)\right| < 0.01$

$\left|x+2\right|\left|x-2\right| < 0.01$

$\left|x-2\right| < \dfrac{0.01}{\left|x+2\right|}$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002$.

So, if $0 < \left|x-2\right| < \delta \approx 0.002$, you have

$\left|x-2\right| < 0.002 = \dfrac{1}{5}(0.01) < \dfrac{1}{\left|x+2\right|}(0.01)$

$\left|x+2\right|\left|x-2\right| < 0.01$

$\left|x^2-4\right| < 0.01$

$\left|\left(x^2-3\right)-1\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|\left(x^2-3\right)-1\right| < 0.005$

$\left|x^2-4\right| < 0.005$

$\left|(x+2)(x-2)\right| < 0.005$

$\left|x+2\right|\left|x-2\right| < 0.005$

$\left|x-2\right| < \dfrac{0.005}{\left|x+2\right|}$

If you assume $1 < x < 3$, then

$\delta = \dfrac{0.005}{5} = 0.001.$

Finally, as in part (a), if $0 < \left|x-2\right| < 0.001$, you have $\left|\left(x^2-3\right)-1\right| < 0.005$.

42. $\lim_{x \to 4} (x^2 + 6) = 4^2 + 6 = 22 = L$

(a) $\left|(x^2 + 6) - 22\right| < 0.01$

$$\left|x^2 - 16\right| < 0.01$$
$$\left|(x + 4)(x - 4)\right| < 0.01$$
$$\left|x + 4\right|\left|x - 4\right| < 0.01$$
$$\left|x - 4\right| < \frac{0.01}{\left|x + 4\right|}$$

If you assume $3 < x < 5$, then

$$\delta = \frac{0.01}{9} \approx 0.00111.$$

So, if $0 < \left|x - 4\right| < \delta \approx \frac{0.01}{9}$, you have

$$\left|x - 4\right| < \frac{0.01}{9} < \frac{0.01}{\left|x + 4\right|}$$
$$\left|(x + 4)(x - 4)\right| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$\left|(x^2 + 6) - 22\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

(b) $\left|(x^2 + 6) - 22\right| < 0.005$

$$\left|x^2 - 16\right| < 0.005$$
$$\left|(x - 4)(x + 4)\right| < 0.005$$
$$\left|x - 4\right|\left|x + 4\right| < 0.005$$
$$\left|x - 4\right| < \frac{0.05}{\left|x + 4\right|}$$

If you assume $3 < x < 5$, then

$$\delta = \frac{0.005}{9} \approx 0.00056.$$

Finally, as in part (a), if $0 < \left|x - 4\right| < \frac{0.005}{9}$, you have $\left|(x^2 + 6) - 22\right| < 0.005$.

43. $\lim_{x \to 4} (x^2 - x) = 16 - 4 = 12 = L$

(a) $\left|(x^2 - x) - 12\right| < 0.01$

$$\left|(x - 4)(x + 3)\right| < 0.01$$
$$\left|x - 4\right|\left|x + 3\right| < 0.01$$
$$\left|x - 4\right| < \frac{0.01}{\left|x + 3\right|}$$

If you assume $3 < x < 5$, then

$$\delta = \frac{0.01}{8} = 0.00125.$$

So, if $0 < \left|x - 4\right| < \frac{0.01}{8}$, you have

$$\left|x - 4\right| < \frac{0.01}{\left|x + 3\right|}$$
$$\left|x - 4\right|\left|x + 3\right| < 0.01$$
$$\left|x^2 - x - 12\right| < 0.01$$
$$\left|(x^2 - x) - 12\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01$$

(b) $\left|(x^2 - x) - 12\right| < 0.005$

$$\left|(x - 4)(x + 3)\right| < 0.005$$
$$\left|x - 4\right|\left|x + 3\right| < 0.005$$
$$\left|x - 4\right| < \frac{0.005}{\left|x + 3\right|}$$

If you assume $3 < x < 5$, then

$$\delta = \frac{0.005}{8} = 0.000625.$$

Finally, as in part (a), if $0 < \left|x - 4\right| < \frac{0.005}{8}$, you have $\left|(x^2 - x) - 12\right| < 0.005$.

44. $\lim_{x\to 3} x^2 = 3^2 = 9 = L$

(a) $$|x^2 - 9| < 0.01$$
$$|(x-3)(x+3)| < 0.01$$
$$|x-3||x+3| < 0.01$$
$$|x-3| < \frac{0.01}{|x+3|}$$

If you assume $2 < x < 4$, then

$\delta = \dfrac{0.01}{7} \approx 0.0014.$

So, if $0 < |x-3| < \dfrac{0.01}{7}$, you have

$$|x-3| < \frac{0.01}{|x+3|}$$
$$|x-3||x+3| < 0.01$$
$$|x^2-9| < 0.01$$
$$|f(x) - L| < 0.01$$

(b) $$|x^2 - 9| < 0.005$$
$$|(x-3)(x+3)| < 0.005$$
$$|x-3||x+3| < 0.005$$
$$|x-3| < \frac{0.005}{|x+3|}$$

If you assume $2 < x < 4$, then

$\delta = \dfrac{0.005}{7} \approx 0.00071.$

Finally, as in part (a), if $0 < |x-3| < \dfrac{0.005}{7}$, you have $|x^2 - 9| < 0.005.$

45. $\lim_{x\to 4}(x+2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$$|(x+2) - 6| < \varepsilon$$
$$|x-4| < \varepsilon = \delta$$

So, let $\delta = \varepsilon$. So, if $0 < |x-4| < \delta = \varepsilon$, you have

$$|x-4| < \varepsilon$$
$$|(x+2)-6| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

46. $\lim_{x\to -2}(4x+5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$$|(4x+5) - (-3)| < \varepsilon$$
$$|4x+8| < \varepsilon$$
$$4|x+2| < \varepsilon$$
$$|x+2| < \frac{\varepsilon}{4} = \delta$$

So, let $\delta = \dfrac{\varepsilon}{4}.$

So, if $0 < |x+2| < \delta = \dfrac{\varepsilon}{4}$, you have

$$|x+2| < \frac{\varepsilon}{4}$$
$$|4x+8| < \varepsilon$$
$$|(4x+5) - (-3)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

47. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}|x - (-4)| < \varepsilon$$
$$|x - (-4)| < 2\varepsilon$$

So, let $\delta = 2\varepsilon$.

So, if $0 < |x - (-4)| < \delta = 2\varepsilon$, you have

$$|x - (-4)| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

48. $\lim_{x\to 3}\left(\frac{3}{4}x + 1\right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\frac{3}{4}x + 1\right) - \frac{13}{4}\right| < \varepsilon$$
$$\left|\frac{3}{4}x - \frac{9}{4}\right| < \varepsilon$$
$$\frac{3}{4}|x - 3| < \varepsilon$$
$$|x - 3| < \frac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < |x - 3| < \delta = \frac{4}{3}\varepsilon$, you have

$$|x - 3| < \frac{4}{3}\varepsilon$$
$$\frac{3}{4}|x - 3| < \varepsilon$$
$$\left|\frac{3}{4}x - \frac{9}{4}\right| < \varepsilon$$
$$\left|\left(\frac{3}{4}x + 1\right) - \frac{13}{4}\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

49. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$|3 - 3| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|3 - 3| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

50. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$: $|-1 - (-1)| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|(-1) - (-1)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

51. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$|x| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0|x - 0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

52. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < |x - 4| < \delta = 3\varepsilon \Rightarrow |x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

53. $\lim_{x\to -5}|x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$: $\left||x - 5| - 10\right| < \varepsilon$

$$|-(x - 5) - 10| < \varepsilon \qquad (x - 5 < 0)$$
$$|-x - 5| < \varepsilon$$
$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$
$$|-(x - 5) - 10| < \varepsilon$$
$$\left||x - 5| - 10\right| < \varepsilon \qquad (\text{because } x - 5 < 0)$$
$$|f(x) - L| < \varepsilon.$$

54. $\lim_{x\to 3} |x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$: $\big||x - 3| - 0\big| < \varepsilon$

$|x - 3| < \varepsilon$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$|x - 3| < \varepsilon$

$\big||x - 3| - 0\big| < \varepsilon$

$|f(x) - L| < \varepsilon.$

55. $\lim_{x\to 1} (x^2 + 1) = 1^2 + 1 = 2$

Given $\varepsilon > 0$: $|(x^2 + 1) - 2| < \varepsilon$

$|x^2 - 1| < \varepsilon$

$|(x + 1)(x - 1)| < \varepsilon$

$|x - 1| < \dfrac{\varepsilon}{|x + 1|}$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have

$|x - 1| < \dfrac{1}{3}\varepsilon < \dfrac{1}{|x + 1|}\varepsilon$

$|x^2 - 1| < \varepsilon$

$|(x^2 + 1) - 2| < \varepsilon$

$|f(x) - 2| < \varepsilon.$

56. $\lim_{x\to -4} (x^2 + 4x) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$: $|(x^2 + 4x) - 0| < \varepsilon$

$|x(x + 4)| < \varepsilon$

$|x + 4| < \dfrac{\varepsilon}{|x|}$

If you assume $-5 < x < -3$, then $\delta = \dfrac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \dfrac{\varepsilon}{5}$, you have

$|x + 4| < \dfrac{\varepsilon}{5} < \dfrac{1}{|x|}\varepsilon$

$|x(x + 4)| < \varepsilon$

$|(x^2 + 4x) - 0| < \varepsilon$

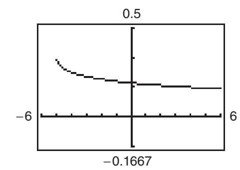
$|f(x) - L| < \varepsilon.$

57. $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} 4 = 4$

58. $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} x = \pi$

59. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

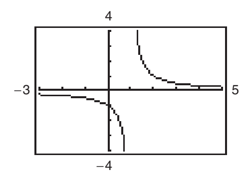
$\lim_{x\to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \dfrac{1}{6}\right)$.

60. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x\to 3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{2}\right)$.

61. $C(t) = 9.99 - 0.79[\![1 - t]\!],\ t > 0$

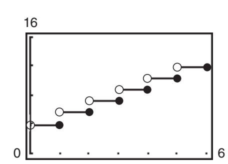
(a) $C(10.75) = 9.99 - 0.79[\![1 - 10.75]\!]$

$= 9.99 - 0.79(-10)$

$= \$17.89$

$C(10.75)$ represents the cost of a 10-minute, 45-second call.

(b)



(c) The limit does not exist because the limits from the left and right are not equal.

62. $C(t) = 5.79 - 0.99[\![1 - t]\!],\ t > 0$

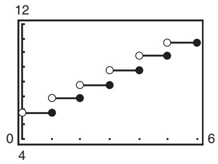
(a) $C(10.75) = 5.79 - 0.99[\![1 - 10.75]\!]$

$= 5.79 - 0.99(-10)$

$= \$15.69$

$C(10.75)$ represents the cost of a 10-minute, 45-second call.

(b)



(c) The limit does not exist because the limits from the left and right are not equal.

63. Choosing a smaller positive value of δ will still satisfy the inequality $|f(x) - L| < \varepsilon$.

64. In the definition of $\lim_{x \to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

65. No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

66. No. The fact that $\lim_{x \to 2} f(x) = 4$ has no bearing on the value of f at 2.

67. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x \to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

68. $V = \frac{4}{3}\pi r^3$, $V = 2.48$

(a) $2.48 = \frac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

$r \approx 0.8397$ in.

(b) $2.45 \le V \le 2.51$

$$2.45 \le \frac{4}{3}\pi r^3 \le 2.51$$

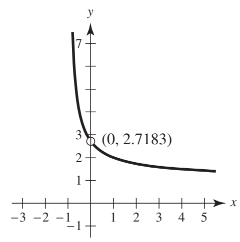
$0.5849 \le r^3 \le 0.5992$

$0.8363 \le r \le 0.8431$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

69. $f(x) = (1 + x)^{1/x}$

$$\lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828$$



x	$f(x)$
–0.1	2.867972
–0.01	2.731999
–0.001	2.719642
–0.0001	2.718418
–0.00001	2.718295
–0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

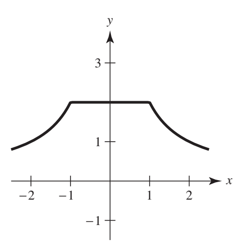
70. $f(x) = \dfrac{|x + 1| - |x - 1|}{x}$

x	–1	–0.5	–0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

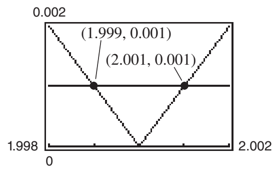
$$\lim_{x \to 0} f(x) = 2$$

Note that for

$$-1 < x < 1,\ x \ne 0,\ f(x) = \frac{(x + 1) + (x - 1)}{x} = 2.$$



71.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

72. (a) $\lim_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

73. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

74. True

75. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$f(2) = 0$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2 \neq 0$

76. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2$ and $f(2) = 0 \neq 2$

77. $f(x) = \sqrt{x}$

$\lim_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

78. $f(x) = \sqrt{x}$

$\lim_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

79. Using a graphing utility, you see that

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

$\lim_{x \to 0} \dfrac{\sin 2x}{x} = 2$, etc.

So, $\lim_{x \to 0} \dfrac{\sin nx}{x} = n$.

80. Using a graphing utility, you see that

$\lim_{x \to 0} \dfrac{\tan x}{x} = 1$

$\lim_{x \to 0} \dfrac{\tan 2x}{x} = 2$, etc.

So, $\lim_{x \to 0} \dfrac{\tan(nx)}{x} = n$.

81. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

82. $f(x) = mx + b, m \neq 0$. Let $\varepsilon > 0$ be given.

Take $\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$

$$|mx - mc| < \varepsilon$$

$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

83. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$|(f(x) - L) - 0| < \varepsilon$.

This means the same as $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

84. (a) $(3x+1)(3x-1)x^2 + 0.01 = (9x^2-1)x^2 + \frac{1}{100}$

$= 9x^4 - x^2 + \frac{1}{100}$

$= \frac{1}{100}(10x^2-1)(90x^2-1)$

So, $(3x+1)(3x-1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\frac{1}{\sqrt{90}}, \frac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in (a, b), the graph is positive.

You can verify this with a graphing utility.

(b) You are given $\lim_{x\to c} g(x) = L > 0$. Let $\varepsilon = \frac{1}{2}L$.

There exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|g(x) - L| < \varepsilon = \frac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$

$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \neq c$, you have $g(x) > \frac{L}{2} > 0$, as desired.

85. The radius OP has a length equal to the altitude z of the triangle plus $\frac{h}{2}$. So, $z = 1 - \frac{h}{2}$.

Area triangle $= \frac{1}{2}b\left(1 - \frac{h}{2}\right)$

Area rectangle $= bh$

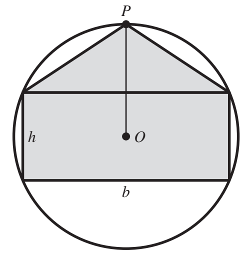
Because these are equal,

$$\frac{1}{2}b\left(1 - \frac{h}{2}\right) = bh$$

$$1 - \frac{h}{2} = 2h$$

$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}.$$



86. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$.

Let x be the length of a side of the cube.

Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

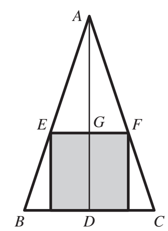
$$\frac{x\sqrt{2}}{2} = \frac{3 - x}{3}$$

Solving for x,

$$3\sqrt{2}x = 6 - 2x$$

$$(3\sqrt{2} + 2)x = 6$$

$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$



Section 1.3 Evaluating Limits Analytically

1. For polynomial functions $p(x)$, substitute c for x, and simplify.

2. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

3. If a function f is squeezed between two functions h and g, $h(x) \le f(x) \le g(x)$, and h and g have the same limit L as $x \to c$, then $\lim_{x\to c} f(x)$ exists and equals L

4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$

5. $\lim_{x\to 2} x^3 = 2^3 = 8$

6. $\lim_{x\to -3} x^4 = (-3)^4 = 81$

7. $\lim_{x\to -3} (2x + 5) = 2(-3) + 5 = -1$

8. $\lim_{x\to 9} (4x - 1) = 4(9) - 1 = 36 - 1 = 35$

9. $\lim_{x\to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

10. $\lim_{x\to 2}(-x^3+1) = (-2)^3 + 1 = -8 + 1 = -7$

11. $\lim_{x\to -3}(2x^2+4x+1) = 2(-3)^2 + 4(-3) + 1$
$= 18 - 12 + 1 = 7$

12. $\lim_{x\to 1}(2x^3 - 6x + 5) = 2(1)^3 - 6(1) + 5$
$= 2 - 6 + 5 = 1$

13. $\lim_{x\to 3}\sqrt{x+8} = \sqrt{3+8} = \sqrt{11}$

14. $\lim_{x\to 2}\sqrt[3]{12x+3} = \sqrt[3]{12(2)+3}$
$= \sqrt[3]{24+3} = \sqrt[3]{27} = 3$

15. $\lim_{x\to -4}(1-x)^3 = [1-(-4)]^3 = 5^3 = 125$

16. $\lim_{x\to 0}(3x-2)^4 = (3(0)-2)^4 = (-2)^4 = 16$

17. $\lim_{x\to 2}\frac{3}{2x+1} = \frac{3}{2(2)+1} = \frac{3}{5}$

18. $\lim_{x\to -5}\frac{5}{x+3} = \frac{5}{-5+3} = -\frac{5}{2}$

19. $\lim_{x\to 1}\frac{x}{x^2+4} = \frac{1}{1^2+4} = \frac{1}{5}$

20. $\lim_{x\to 1}\frac{3x+5}{x+1} = \frac{3(1)+5}{1+1} = \frac{3+5}{2} = \frac{8}{2} = 4$

21. $\lim_{x\to 7}\frac{3x}{\sqrt{x+2}} = \frac{3(7)}{\sqrt{7+2}} = \frac{21}{3} = 7$

22. $\lim_{x\to 3}\frac{\sqrt{x+6}}{x+2} = \frac{\sqrt{3+6}}{3+2} = \frac{\sqrt{9}}{5} = \frac{3}{5}$

23. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

24. (a) $\lim_{x\to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x\to 4} g(x) = 4^2 = 16$

(c) $\lim_{x\to -3} g(f(x)) = g(4) = 16$

25. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

26. (a) $\lim_{x\to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x\to 21} g(x) = \sqrt[3]{21+6} = 3$

(c) $\lim_{x\to 4} g(f(x)) = g(21) = 3$

27. $\lim_{x\to \pi/2}\sin x = \sin\frac{\pi}{2} = 1$

28. $\lim_{x\to \pi}\tan x = \tan\pi = 0$

29. $\lim_{x\to 1}\cos\frac{\pi x}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$

30. $\lim_{x\to 2}\sin\frac{\pi x}{12} = \sin\frac{\pi(2)}{12} = \sin\frac{\pi}{6} = \frac{1}{2}$

31. $\lim_{x\to 0}\sec 2x = \sec 0 = 1$

32. $\lim_{x\to \pi}\cos 3x = \cos 3\pi = -1$

33. $\lim_{x\to 5\pi/6}\sin x = \sin\frac{5\pi}{6} = \frac{1}{2}$

34. $\lim_{x\to 5\pi/3}\cos x = \cos\frac{5\pi}{3} = \frac{1}{2}$

35. $\lim_{x\to 3}\tan\left(\frac{\pi x}{4}\right) = \tan\frac{3\pi}{4} = -1$

36. $\lim_{x\to 7}\sec\left(\frac{\pi x}{6}\right) = \sec\frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

37. $\lim_{x\to c} f(x) = \frac{2}{5}, \lim_{x\to c} g(x) = 2$

(a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c} g(x) = 5(2) = 10$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \frac{2}{5} + 2 = \frac{12}{5}$

(c) $\lim_{x\to c}[f(x) + g(x)] = \left[\lim_{x\to c} f(x)\right] + \left[\lim_{x\to c} g(x)\right] = \frac{2}{5}(2) = \frac{4}{5}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2/5}{2} = \frac{1}{5}$

38. $\lim_{x\to c} f(x) = 2,\ \lim_{x\to c} g(x) = \frac{3}{4}$

(a) $\lim_{x\to c}\left[4f(x)\right] = 4\lim_{x\to c} f(x) = 4(2) = 8$

(b) $\lim_{x\to c}\left[f(x) + g(x)\right] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$

(c) $\lim_{x\to c}\left[f(x)g(x)\right] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$

39. $\lim_{x\to c} f(x) = 16$

(a) $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (16)^2 = 256$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{16} = 4$

(c) $\lim_{x\to c}\left[3f(x)\right] = 3\left[\lim_{x\to c} f(x)\right] = 3(16) = 48$

(d) $\lim_{x\to c}\left[f(x)\right]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (16)^{3/2} = 64$

40. $\lim_{x\to c} f(x) = 27$

(a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

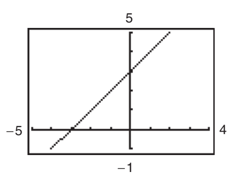
(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}\left[f(x)\right]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

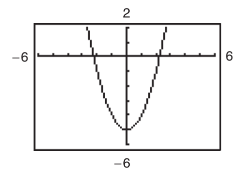
41. $f(x) = \frac{x^2 + 3x}{x} = \frac{x(x + 3)}{x}$ and $g(x) = x + 3$ agree except at $x = 0$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0}(x + 3) = 0 + 3 = 3$



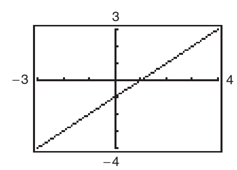
42. $f(x) = \frac{x^4 - 5x^2}{x^2} = \frac{x^2(x^2 - 5)}{x^2}$ and $g(x) = x^2 - 5$ agree except at $x = 0$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0}(x^2 - 5) = 0^2 - 5 = -5$



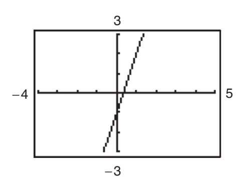
43. $f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1}(x - 1) = -1 - 1 = -2$



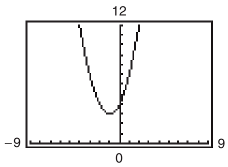
44. $f(x) = \frac{3x^2 + 5x - 2}{x + 2} = \frac{(x + 2)(3x - 1)}{x + 2}$ and $g(x) = 3x - 1$ agree except at $x = -2$.

$\lim_{x\to -2} f(x) = \lim_{x\to -2} g(x) = \lim_{x\to -2}(3x - 1)$
$= 3(-2) - 1 = -7$



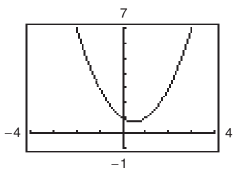
45. $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2} \left(x^2 + 2x + 4\right) = 2^2 + 2(2) + 4 = 12$$



46. $f(x) = \dfrac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1} \left(x^2 - x + 1\right) = (-1)^2 - (-1) + 1 = 3$$



47. $\displaystyle \lim_{x\to 0} \frac{x}{x^2 - x} = \lim_{x\to 0} \frac{x}{x(x-1)} = \lim_{x\to 0} \frac{1}{x-1} = \frac{1}{0-1} = -1$

48. $\displaystyle \lim_{x\to 0} \frac{7x^3 - x^2}{x} = \lim_{x\to 0} \left(7x^2 - x\right) = 0 - 0 = 0$

49. $\displaystyle \lim_{x\to 4} \frac{x - 4}{x^2 - 16} = \lim_{x\to 4} \frac{x - 4}{(x + 4)(x - 4)} = \lim_{x\to 4} \frac{1}{x + 4} = \frac{1}{4 + 4} = \frac{1}{8}$

50. $\displaystyle \lim_{x\to 5} \frac{5 - x}{x^2 - 25} = \lim_{x\to 5} \frac{-(x - 5)}{(x - 5)(x + 5)} = \lim_{x\to 5} \frac{-1}{x + 5} = \frac{-1}{5 + 5} = -\frac{1}{10}$

51. $\displaystyle \lim_{x\to -3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} = \lim_{x\to -3} \frac{x - 2}{x - 3} = \frac{-3 - 2}{-3 - 3} = \frac{-5}{-6} = \frac{5}{6}$

52. $\displaystyle \lim_{x\to 2} \frac{x^2 + 2x - 8}{x^2 - x - 2} = \lim_{x\to 2} \frac{(x - 2)(x + 4)}{(x - 2)(x + 1)} = \lim_{x\to 2} \frac{x + 4}{x + 1} = \frac{2 + 4}{2 + 1} = \frac{6}{3} = 2$

53. $\displaystyle \lim_{x\to 4} \frac{\sqrt{x + 5} - 3}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x + 5} - 3}{x - 4} \cdot \frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3}$

$\displaystyle = \lim_{x\to 4} \frac{(x + 5) - 9}{(x - 4)\left(\sqrt{x + 5} + 3\right)} = \lim_{x\to 4} \frac{1}{\sqrt{x + 5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$

54. $\displaystyle \lim_{x\to 3} \frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x\to 3} \frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x\to 3} \frac{x - 3}{(x - 3)\left[\sqrt{x + 1} + 2\right]}$

$\displaystyle = \lim_{x\to 3} \frac{1}{\sqrt{x + 1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$

55. $\displaystyle \lim_{x\to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} = \lim_{x\to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x + 5} + \sqrt{5}}{\sqrt{x + 5} + \sqrt{5}}$

$\displaystyle = \lim_{x\to 0} \frac{(x + 5) - 5}{x\left(\sqrt{x + 5} + \sqrt{5}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x + 5} + \sqrt{5}} = \frac{1}{\sqrt{5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

56. $\displaystyle \lim_{x\to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$

$\displaystyle = \lim_{x\to 0} \frac{2 + x - 2}{\left(\sqrt{2 + x} + \sqrt{2}\right)x} = \lim_{x\to 0} \frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

57. $\lim_{x\to 0} \dfrac{\dfrac{1}{3+x} - \dfrac{1}{3}}{x} = \lim_{x\to 0} \dfrac{3-(3+x)}{(3+x)3(x)} = \lim_{x\to 0} \dfrac{-x}{(3+x)(3)(x)} = \lim_{x\to 0} \dfrac{-1}{(3+x)3} = \dfrac{-1}{(3)3} = -\dfrac{1}{9}$

58. $\lim_{x\to 0} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x} = \lim_{x\to 0} \dfrac{\dfrac{4-(x+4)}{4(x+4)}}{x}$

$= \lim_{x\to 0} \dfrac{-1}{4(x+4)} = \dfrac{-1}{4(4)} = -\dfrac{1}{16}$

59. $\lim_{\Delta x\to 0} \dfrac{2(x+\Delta x)-2x}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{2x+2\Delta x-2x}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$

60. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x+\Delta x) = 2x$

61. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2 - 2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$= \lim_{\Delta x\to 0}(2x+\Delta x-2) = 2x-2$

62. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$

$= \lim_{\Delta x\to 0} \dfrac{\Delta x\left(3x^2+3x\Delta x+(\Delta x)^2\right)}{\Delta x} = \lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right) = 3x^2$

63. $\lim_{x\to 0} \dfrac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\dfrac{\sin x}{x}\right)\left(\dfrac{1}{5}\right)\right] = (1)\left(\dfrac{1}{5}\right) = \dfrac{1}{5}$

64. $\lim_{x\to 0} \dfrac{3(1-\cos x)}{x} = \lim_{x\to 0}\left[3\left(\dfrac{(1-\cos x)}{x}\right)\right] = (3)(0) = 0$

65. $\lim_{x\to 0} \dfrac{(\sin x)(1-\cos x)}{x^2} = \lim_{x\to 0}\left[\dfrac{\sin x}{x}\cdot\dfrac{1-\cos x}{x}\right]$

$= (1)(0) = 0$

66. $\lim_{\theta\to 0} \dfrac{\cos\theta\tan\theta}{\theta} = \lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

67. $\lim_{x\to 0} \dfrac{\sin^2 x}{x} = \lim_{x\to 0}\left[\dfrac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

68. $\lim_{x\to 0} \dfrac{\tan^2 x}{x} = \lim_{x\to 0}\dfrac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0}\left[\dfrac{\sin x}{x}\cdot\dfrac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

69. $\lim_{h\to 0} \dfrac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\dfrac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

70. $\lim_{\phi\to\pi} \phi\sec\phi = \pi(-1) = -\pi$

71. $\lim_{x\to 0} \dfrac{6-6\cos x}{3} = \dfrac{6-6\cos 0}{3} = \dfrac{6-6}{3} = 0$

72. $\lim_{x\to 0} \dfrac{\cos x-\sin x-1}{2x} = \lim_{x\to 0}\dfrac{-\sin x}{2x} + \lim_{x\to 0}\dfrac{\cos x-1}{2x}$

$= -\dfrac{1}{2}\lim_{x\to 0}\dfrac{\sin x}{x} - \dfrac{1}{2}\lim_{x\to 0}\dfrac{1-\cos x}{x}$

$= -\dfrac{1}{2}(1) - \dfrac{1}{2}(0) = -\dfrac{1}{2}$

73. $\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

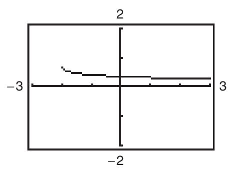
74. $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right]$

$= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

75. $f(x) = \frac{\sqrt{x+2}-\sqrt{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at $x = 0$.

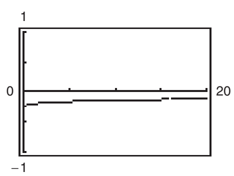
Analytically, $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$= \lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$

76. $f(x) = \frac{4-\sqrt{x}}{x-16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	–0.1252	–0.125	–0.125	?	–0.125	–0.125	–0.1248

It appears that the limit is –0.125.



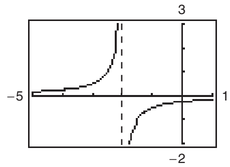
The graph has a hole at $x = 16$.

Analytically, $\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16} = \lim_{x\to 16}\frac{\left(4-\sqrt{x}\right)}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)} = \lim_{x\to 16}\frac{-1}{\sqrt{x}+4} = -\frac{1}{8}.$

77. $f(x) = \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.263	–0.251	–0.250	?	–0.250	–0.249	–0.238

It appears that the limit is –0.250.



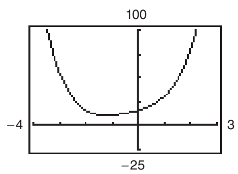
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$

78. $f(x) = \dfrac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at $x = 2$.

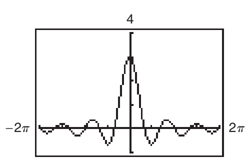
Analytically, $\displaystyle\lim_{x\to 2} \frac{x^5 - 32}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim_{x\to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

79. $f(t) = \dfrac{\sin 3t}{t}$

t	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



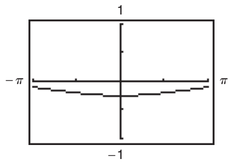
The graph has a hole at $t = 0$.

Analytically, $\displaystyle\lim_{t\to 0} \frac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\frac{\sin 3t}{3t}\right) = 3(1) = 3.$

80. $f(x) = \dfrac{\cos x - 1}{2x^2}$

x	–1	–0.1	–0.01	0.01	0.1	1
$f(x)$	–0.2298	–0.2498	–0.25	–0.25	–0.2498	–0.2298

It appears that the limit is –0.25.



The graph has a hole at $x = 0$.

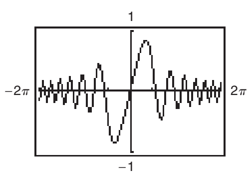
Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{2x^2(\cos x + 1)} = \dfrac{-\sin^2 x}{2x^2(\cos x + 1)} = \dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}$

$$\lim_{x \to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

81. $f(x) = \dfrac{\sin x^2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.099998	–0.01	–0.001	?	0.001	0.01	0.099998

It appears that the limit is 0.



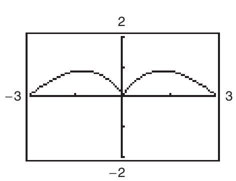
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} x\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0.$

82. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

It appears that the limit is 0.



The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x \to 0} \frac{\sin x}{\sqrt[3]{x}} = \lim_{x \to 0} \sqrt[3]{x^2}\left(\frac{\sin x}{x}\right) = (0)(1) = 0.$

83. $f(x) = 3x - 2$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3$$

84. $f(x) = -6x + 3$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[-6(x + \Delta x) + 3] - [-6x + 3]}{\Delta x} = \lim_{\Delta x \to 0} \frac{-6x - 6\Delta x + 3 + 6x - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-6\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (-6) = -6$$

85. $f(x) = x^2 - 4x$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$$

86. $f(x) = 3x^2 + 1$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[3(x + \Delta x)^2 + 1\right] - \left[3x^2 + 1\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{(3x^2 + 6x\Delta x + 1) - (3x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 6x = 6x$$

87. $f(x) = 2\sqrt{x}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sqrt{x + \Delta x} - 2\sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\left(\sqrt{x + \Delta x} - \sqrt{x}\right)}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{2(x + \Delta x - x)}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{2}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} = x^{-1/2}$$

88. $f(x) = \sqrt{x} - 5$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x} - 5\right) - \left(\sqrt{x} - 5\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

89. $f(x) = \dfrac{1}{x + 3}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + 3} - \dfrac{1}{x + 3}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}$$

90. $f(x) = \dfrac{1}{x^2}$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 - (x+\Delta x)^2}{x^2(x+\Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - \left[x^2 + 2x\Delta x + (\Delta x)^2\right]}{x^2(x+\Delta x)^2 \Delta x} = \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{x^2(x+\Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{x^2(x+\Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

91. $\lim_{x\to 0}\left(4 - x^2\right) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0}\left(3 + x^2\right)$

$$4 \le \lim_{x\to 0} f(x) \le 4$$

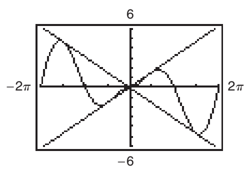
Therefore, $\lim_{x\to 0} f(x) = 4$.

92. $\lim_{x\to a}\left[b - |x - a|\right] \le \lim_{x\to a} f(x) \le \lim_{x\to a}\left[b + |x - a|\right]$

$$b \le \lim_{x\to a} f(x) \le b$$

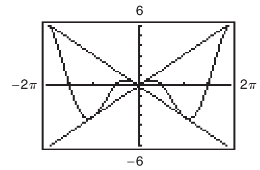
Therefore, $\lim_{x\to a} f(x) = b$.

93. $f(x) = |x|\sin x$



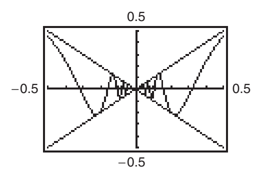
$\lim_{x\to 0} |x|\sin x = 0$

94. $f(x) = |x|\cos x$



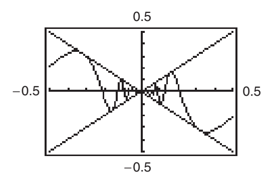
$\lim_{x\to 0} |x|\cos x = 0$

95. $f(x) = x \sin \dfrac{1}{x}$



$\lim_{x\to 0}\left(x \sin \dfrac{1}{x}\right) = 0$

96. $f(x) = x \cos \dfrac{1}{x}$



$\lim_{x\to 0}\left(x \cos \dfrac{1}{x}\right) = 0$

97. (a) Two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

(b) $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

(Other answers possible.)

98. Answers will vary. *Sample answers:*

(a) linear: $f(x) = \frac{1}{2}x$; $\lim_{x\to 8} \frac{1}{2}x = \frac{1}{2}(8) = 4$

(b) polynomial of degree 2: $f(x) = x^2 - 60$; $\lim_{x\to 8}\left(x^2 - 60\right) = 8^2 - 60 = 4$

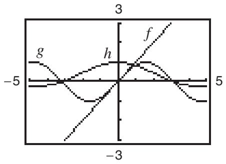
(c) rational: $f(x) = \dfrac{x}{2x - 14}$; $\lim_{x\to 8} \dfrac{x}{2x - 14} = \dfrac{8}{2(8) - 14} = \dfrac{8}{2} = 4$

(d) radical: $f(x) = \sqrt{x + 8}$; $\lim_{x\to 8} \sqrt{x + 8} = \sqrt{8 + 8} = \sqrt{16} = 4$

(e) cosine: $f(x) = 4\cos(\pi x)$; $\lim_{x\to 8} 4\cos(\pi x) = 4\cos 8\pi = 4(1) = 4$

(f) sine: $f(x) = 4\sin\left(\dfrac{\pi}{16}x\right)$; $\lim_{x\to 8} 4\sin\left(\dfrac{\pi}{16}x\right) = 4\sin\dfrac{\pi}{2} = 4(1) = 4$

99. $f(x) = x, g(x) = \sin x, h(x) = \dfrac{\sin x}{x}$



When the x-values are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. So, $|g|/|f| \approx 1$ when x is "close to" 0.

100. (a) Use the dividing out technique because the numerator and denominator have a common factor.

(b) Use the rationalizing technique because the numerator involves a radical expression.

101. $s(t) = -16t^2 + 500$

$$\lim_{t\to 2}\frac{s(2)-s(t)}{2-t} = \lim_{t\to 2}\frac{-16(2)^2 + 500 - \left(-16t^2 + 500\right)}{2-t}$$
$$= \lim_{t\to 2}\frac{436 + 16t^2 - 500}{2-t}$$
$$= \lim_{t\to 2}\frac{16\left(t^2 - 4\right)}{2-t}$$
$$= \lim_{t\to 2}\frac{16(t-2)(t+2)}{2-t}$$
$$= \lim_{t\to 2} -16(t+2) = -64 \text{ ft/sec}$$

The paint can is falling at about 64 feet/second.

102. $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\dfrac{500}{16}} = \dfrac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \dfrac{5\sqrt{5}}{2}$ is

$$\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{0 - \left(-16t^2 + 500\right)}{\frac{5\sqrt{5}}{2} - t}$$
$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t}$$
$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t}$$
$$= \lim_{t\to\frac{5\sqrt{5}}{2}}\left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec}$$
$$\approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

103. $s(t) = -4.9t^2 + 200$

$$\lim_{t\to 3}\frac{s(3)-s(t)}{3-t} = \lim_{t\to 3}\frac{-4.9(3)^2 + 200 - \left(-4.9t^2 + 200\right)}{3-t}$$
$$= \lim_{t\to 3}\frac{4.9\left(t^2 - 9\right)}{3-t}$$
$$= \lim_{t\to 3}\frac{4.9(t-3)(t+3)}{3-t}$$
$$= \lim_{t\to 3}\left[-4.9(t+3)\right]$$
$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

104. $-4.9t^2 + 200 = 0$ when $t = \sqrt{\dfrac{200}{4.9}} = \dfrac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \dfrac{20\sqrt{5}}{7}$ is

$$\begin{aligned}\lim_{t\to a}\frac{s(a)-s(t)}{a-t} &= \lim_{t\to a}\frac{0-\left[-4.9t^2+200\right]}{a-t}\\ &= \lim_{t\to a}\frac{4.9(t+a)(t-a)}{a-t}\\ &= \lim_{t\to\frac{20\sqrt{5}}{7}}\left[-4.9\left(t+\frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5}\text{ m/sec}\\ &\approx -62.6\text{ m/sec.}\end{aligned}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ do not exist. However,

$$\lim_{x\to0}\left[f(x)+g(x)\right] = \lim_{x\to0}\left[\frac{1}{x}+\left(-\frac{1}{x}\right)\right] = \lim_{x\to0}[0] = 0 \text{ and therefore does not exist.}$$

106. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}\left[f(x)+g(x)\right]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

107. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|f(x) - b\right| < \varepsilon$ whenever $\left|x - c\right| < \delta$. Because $\left|f(x)-b\right| = \left|b-b\right| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

108. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

109. If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $\left|f(x) - L\right| < \varepsilon/\left|b\right|$ whenever $0 < \left|x-c\right| < \delta$. So, whenever $0 < \left|x-c\right| < \delta$, we have

$$\left|b\right|\left|f(x)-L\right| < \varepsilon \quad\text{or}\quad \left|bf(x)-bL\right| < \varepsilon$$

which implies that $\lim_{x\to c}\left[bf(x)\right] = bL$.

110. Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x) - 0\right| < \varepsilon$ whenever $0 < \left|x - c\right| < \delta$.

Now $\left|f(x)-0\right| = \left|f(x)\right| = \left|\left|f(x)\right| - 0\right| < \varepsilon$ for $\left|x-c\right| < \delta$. Therefore, $\lim_{x\to c}\left|f(x)\right| = 0$.

111.

$$\begin{aligned}-M\left|f(x)\right| &\le f(x)g(x) \le M\left|f(x)\right|\\ \lim_{x\to c}\left(-M\left|f(x)\right|\right) &\le \lim_{x\to c}\left[f(x)g(x)\right] \le \lim_{x\to c}\left(M\left|f(x)\right|\right)\\ -M(0) &\le \lim_{x\to c}\left[f(x)g(x)\right] \le M(0)\\ 0 &\le \lim_{x\to c}\left[f(x)g(x)\right] \le 0\end{aligned}$$

Therefore, $\lim_{x\to c}\left[f(x)g(x)\right] = 0$.

112. (a) If $\lim_{x\to c}\left|f(x)\right| = 0$, then $\lim_{x\to c}\left[-\left|f(x)\right|\right] = 0$.

$$\begin{aligned}-\left|f(x)\right| &\le f(x) \le \left|f(x)\right|\\ \lim_{x\to c}\left[-\left|f(x)\right|\right] &\le \lim_{x\to c} f(x) \le \lim_{x\to c}\left|f(x)\right|\\ 0 &\le \lim_{x\to c} f(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x) - L\right| < \varepsilon$ whenever $0 < \left|x-c\right| < \delta$. Since $\left|\left|f(x)\right| - \left|L\right|\right| \le \left|f(x) - L\right| < \varepsilon$ for $\left|x - c\right| < \delta$, then $\lim_{x\to c}\left|f(x)\right| = \left|L\right|$.