Chapter 2

Problem 2.1

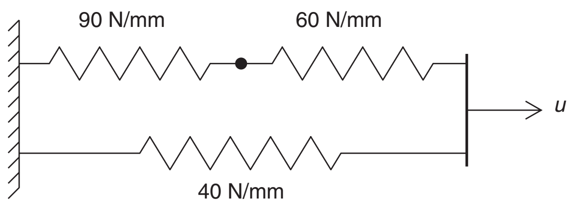

Figure S2.1

Referring to Figure S2.1 the springs with stiffness 60 N/mm and 90 N/mm are placed in series and have an effective stiffness given by

$$k_1 = \frac{1}{1/60 + 1/90} = 36 \text{ N/mm}$$

This combination is now placed in parallel with the spring of stiffness 40 N/mm giving a final effective stiffness of

$$k_{\text{eff}} = k_1 + 40 = 76 \text{ N/mm}$$

Problem 2.2

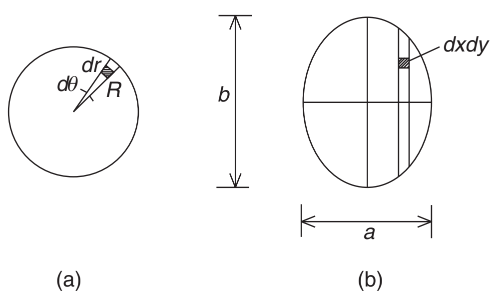

Figure S2.2

The infinitesimal area shown in Figure S2.2(a) is equal to $rd\theta dr$. When the circular disc moves in the x direction with acceleration $\ddot{u}_x$ the inertia force on the infinitesimal are is $\gamma rd\theta dr\ddot{u}_x$, where γ ids the mass per unit area. The resultant inertia force on the disc acting in the negative x direction is given by

$$I_x = \int_0^R \int_0^{2\pi} \gamma \ddot{u}_x r d\theta dr = \gamma \pi R^2 \ddot{u}_x = M\ddot{u}_x$$

where M is the total mass of the disc. The resultant moment of the inertia forces about the centre of the disc, which is also the centre of mass is given by

$$M_x = \int_0^R \int_0^{2\pi} \gamma \ddot{u}_x r^2 \sin\theta d\theta dr = 0$$

In a similar manner we get

$$I_y = M\ddot{u}_y$$

For an angular acceleration $\ddot{\theta}$ about the center of mass the inertia force on the infinitesimal element is directed along the tangent and is $\gamma r^2\ddot{\theta}d\theta dr$. The x component of this force is $\gamma r^2\ddot{\theta}d\theta dr \sin\theta$. It is easily seen that the resultant of all x direction forces is zero. In a similar manner the resultant y direction force is zero. However, a clockwise moment about the center of the disc exists and is given by

$$M_\theta = \int_0^R \int_0^{2\pi} \gamma\ddot{\theta} r^3 d\theta dr = \gamma\pi R^2 \frac{R^2}{2}\ddot{\theta} = M\frac{R^2}{2}\ddot{\theta}$$

The elliptical plate shown in Figure S2.2(c) is divided into the infinitesimal elements as shown. The mass of an element is $\gamma dxdy$ and the inertia force acting on it when the disc undergoes translation in the x direction with acceleration $\ddot{u}_x$ is $\gamma\ddot{u}_x dxdy$. The resultant inertia force in the negative x direction is given by

$$\begin{aligned} I_x &= \int_{-a/2}^{a/2} \int_{-b/2\sqrt{1-4x^2/a^2}}^{b/2\sqrt{1-4x^2/a^2}} \gamma\ddot{u}_y dydx \\ &= \gamma\ddot{u}_x \int_{-a/2}^{a/2} b\sqrt{1-4x^2/a^2}dx \\ &= \frac{\pi\gamma ab}{4} = M\ddot{u}_x \end{aligned}$$

The moment of the x direction inertia force on an element is $\gamma\ddot{u}_x ydxdy$. The resultant moment obtained over the area is zero. The inertia force produced by an acceleration in the y direction is obtained in a similar manner and is $M\ddot{u}_y$ directed in the negative y direction.

An angular acceleration $\ddot{\theta}$ produces a clockwise moment equal to $\gamma r^2\ddot{\theta}dxdy = \gamma\left(x^2+y^2\right)\ddot{\theta}dxdy$. Integration over the area yields the resultant moment, which is clockwise

$$\begin{aligned} I_\theta &= \int_{-a/2}^{a/2} \int_{-b/2\sqrt{1-4x^2/a^2}}^{b/2\sqrt{1-4x^2/a^2}} \gamma\ddot{\theta}\left(x^2+y^2\right) dydx \\ &= \gamma\frac{\pi ab}{4}\frac{a^2+b^2}{16}\ddot{\theta} = M\frac{a^2+b^2}{16}\ddot{\theta} \end{aligned}$$

The x and y direction inertia forces produced on the infinitesimal element are $-\gamma\ddot{\theta}\sin\theta dxdy$ and $\gamma\ddot{\theta}\cos\theta dxdy$, respectively. When summed over the area the net forces produced by these are easily shown to be zero.

Problem 2.3

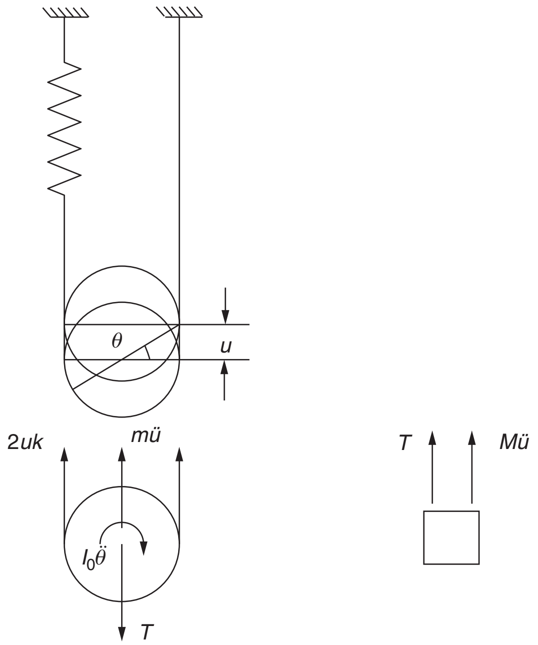

Figure S2.3

As the center of the pulley moves a distance u, the spring is stretched by $2u$ and the pulley rotates through u/R. Referring to the free body diagrams shown in Figure S2.3, the conditions of equilibrium are

$$2uk \times 2R + m\ddot{u}R + I_0\ddot{\theta} - TR = 0$$
$$T + M\ddot{u} = 0$$

Substituting $I_0 = mR^2/2$ and eliminating T gives

$$\left(M + \frac{3m}{2}\right)\ddot{u} + 4ku = 0$$

Problem 2.4

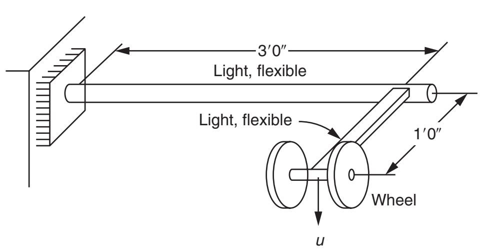

Figure S2.4

$$I_b = \frac{1}{12} \times 1 \times \left(\frac{1}{4}\right)^3 = \frac{1}{768}$$
$$I_s = \frac{\pi}{64}\ , \quad J_s = \frac{\pi}{32}$$

The displacement at location A, shown in Figure S2.4, due to a unit load is given by

$$\begin{aligned}\Delta_A &= \frac{L_b^3}{3EI_b} + \frac{L_s^3}{3EI_s} + \frac{L_b^2 L_s}{GJ_s} \\ &= \frac{12^3 \times 768}{3 \times 30 \times 10^6} + \frac{36^3 \times 64}{3 \times 30 \times 10^6 \times \pi} \\ &\quad + \frac{12^2 \times 36 \times 32}{12 \times 10^6 \times \pi} \\ &= 0.01475 + 0.01056 + 0.0044 \\ &= 0.0297 \text{ in.}\end{aligned}$$

Equivalent spring stiffness $= \frac{1}{0.0297} = 33.67\ \frac{\text{lb.}}{\text{in.}}$
Mass $= \frac{10}{386.4} = 0.02588\ \frac{\text{lb.s}^2}{\text{in.}}$
The equation of motion is given by

$$0.02588\ddot{u} + 33.67u = 0$$

or

$$\ddot{u} + 1300u = 0$$

Problem 2.5

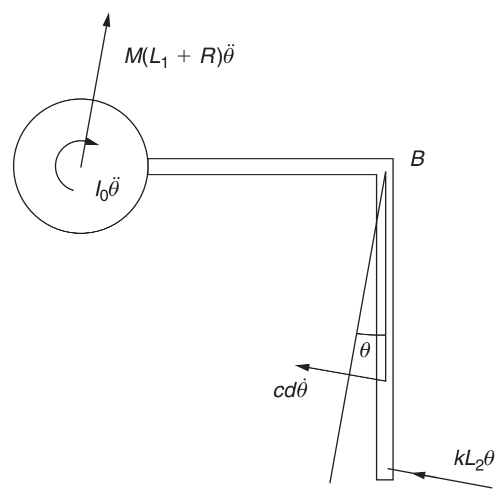

Figure S2.5

Taking moments about B shown in Figure S2.5, and noting that $I_0 = MR^2/2$

$$M\left(L_1 + R\right)^2\ddot{\theta} + \frac{MR^2}{2}\ddot{\theta} + cd^2\dot{\theta} + kL_2^2\theta = 0$$

or

$$M\left(L_1^2 + \frac{3}{2}R^2 + 2L_1R\right)\ddot{\theta} + cd^2\dot{\theta} + kL_2^2\theta = 0$$

Problem 2.6

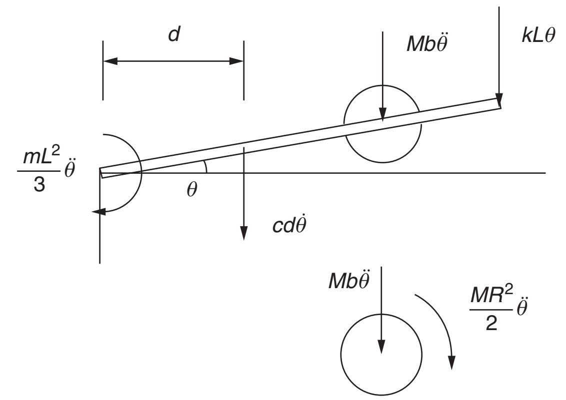

Figure S2.6

Taking moments about A shown in Figure S2.6

$$\left(\frac{mL^2}{3} + Mb^2\right)\ddot{\theta} + cd^2\dot{\theta} + kL^2\theta = 0$$

When the flywheel is braked, an additional rotational inertia term will exist. Moment equilibrium gives

$$\left[\frac{mL^2}{3} + M\left(b^2 + \frac{R^2}{2}\right)\right]\ddot{\theta} + cd^2\dot{\theta} + kL^2\theta = 0$$

Problem 2.7

Boundary conditions

$$\left.\begin{aligned}\psi &= 0\\ \frac{d\psi}{dx} &= 0\end{aligned}\right\} x = 0$$

$$\left.\begin{aligned}\psi &= 0\\ EI\frac{d^2\psi}{dx^2} &= 0\end{aligned}\right\} x = L$$

where

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi} \times \frac{1}{L} = \frac{1}{L}\left(8\xi^3 - 15\xi^2 + 6\xi\right)$$
$$\frac{d^2\psi}{dx^2} = \frac{1}{L^2}\left(24\xi^2 - 30\xi + 6\right)$$

All four boundary conditions are satisfied

$$\begin{aligned}m^* &= \int_0^L \overline{m}\left\{\psi(x)\right\}^2 dx\\ &= \overline{m}L\int_0^1 \left\{\psi(\xi)\right\}^2 d\xi = 0.03016\overline{m}L\\ k^* &= \int_0^L EI\left\{\frac{d^2\psi(x)}{dx^2}\right\}^2 dx\\ &= \frac{EI}{L^3}\int_0^1 \left\{\frac{d^2\psi(\xi)}{d\xi^2}\right\}^2 d\xi = 7.2\frac{EI}{L^3}\end{aligned}$$

Problem 2.8

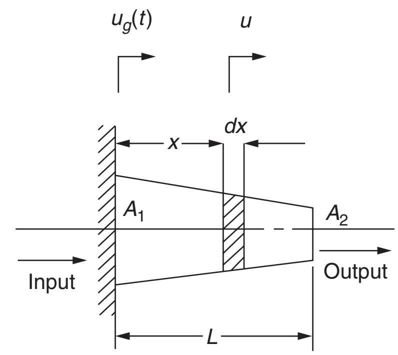

Figure S2.8

$$\frac{d\xi}{dx} = \frac{1}{L}, \quad A(\xi) = A_1 + (A_2 - A_1)\,\xi$$
$$\frac{d\psi}{dx} = (2 - 2\xi)\,\frac{1}{L}, \quad \overline{m}(\xi) = \rho A(\xi)$$

where ρ is the mass density.

$$\begin{aligned}m^* &= \int_0^L \overline{m}(x)\left\{\psi(x)\right\}^2 dx\\ &= \int_0^1 \rho L\left\{A_1 + (A_2 - A_1)\,\xi\right\}\\ &\qquad \left(4\xi^2 + \xi^4 - 4\xi^3\right) d\xi\\ &= \frac{\rho L}{30}\left(5A_1 + 11A_2\right)\\ k^* &= \int_0^L EA(x)\left\{\frac{d\psi(x)}{dx}\right\}^2 dx\\ &= \frac{E}{L}\int_0^1 \left\{A_1 + (A_2 - A_1)\,\xi\right\}\\ &\qquad \left(4 + 4\xi^2 - 8\xi\right) d\xi\\ &= \frac{E}{L}\left(A_1 + \frac{1}{3}A_2\right)\\ p^* &= -\int_0^L \overline{m}(x)\ddot{u}_g\psi(x)dx\\ &= -\int_0^1 \rho L\ddot{u}_g\left\{A_1 + (A_2 - A_1)\,\xi\right\}\left(2\xi - \xi^2\right) d\xi\\ &= -\rho L\ddot{u}_g\left(\frac{1}{4}A_1 + \frac{5}{12}A_2\right)\end{aligned}$$

Problem 2.9

Refer to Figure S2.9 where the inertia force on an infinitesimal element is shown as $\bar{m}\frac{\partial^2 \ddot{u}^t}{\partial x^2}dx$. The total displacement $u^t = u + u_g$, where u is the displacement relative to the ground and u_g is the displacement of the ground. The inertia force thus becomes $\bar{m}\left(\ddot{z}(t)\psi(x)+\ddot{u}_g\right)dx$. The first term within the parentheses produces the generalized mass m^* while the second term produces the generalized force p^* as follows

$$m^* = \int_0^L \bar{m}[\psi(x)]^2 dx = \frac{33}{140}\bar{m}L$$

$$p^* = -\bar{m}\ddot{u}_g \int_0^L \psi(x)dx = -\frac{3}{8}\bar{m}L\ddot{u}_g$$

The generalized stiffness is given by

$$k^* = \int_0^L EI\left[\psi''(x)\right]^2 dx = \frac{3EI}{L^3}$$

The geometric stiffness produced by the axial force $S(x) = (L-x)\bar{m}g$ is obtained from

$$k_G = \int_0^L S(x)\left[\psi'(x)\right]^2 dx = \frac{3}{8}\bar{m}g$$

The equation of motion including the gravity effect is

$$\frac{33}{140}\bar{m}L\ddot{z} + \left(\frac{3EI}{L^3} - \frac{3}{8}\bar{m}g\right)z = -\frac{3}{8}\bar{m}Lg$$

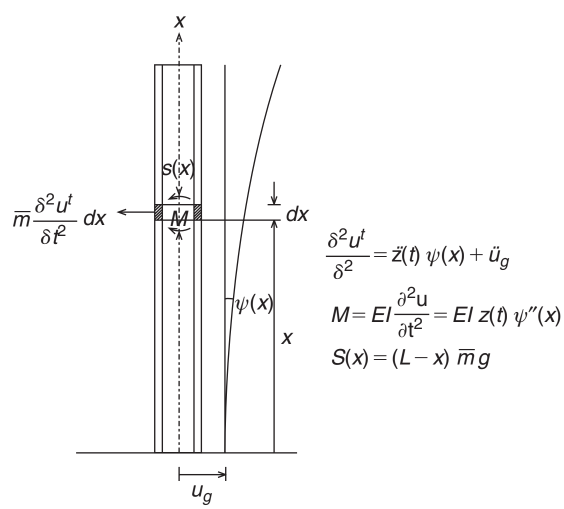

Figure S2.9

Problem 2.10

$$u = z\psi(x)$$

$$\psi = \sin\frac{\pi x}{L}, \qquad \frac{d^2\psi}{dx^2} = -\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

$$m^* = \int_0^L \overline{m}\sin^2\frac{\pi x}{L}dx = \frac{\overline{m}L}{2}$$

$$k^* = \int_0^L EI\frac{\pi^4}{L^4}\sin^2\frac{\pi x}{L}dx = \frac{\pi^4 EI}{2L^3}$$

$$p^* = \int_0^L F\delta(x-vt)\sin\frac{\pi x}{L}dx$$

$$= F\sin\frac{\pi vt}{L}$$

Equation of motion:

$$\frac{\overline{m}L}{2}\ddot{z} + \frac{\pi^4 EI}{2L^3}z = F\sin\frac{\pi vt}{L}$$

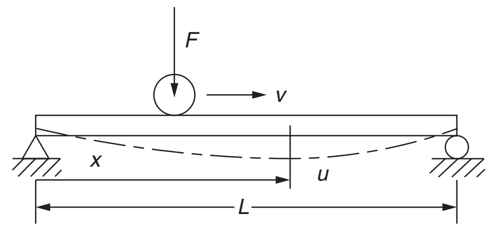

Figure S2.10

Problem 2.11

The mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are given by

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ kip s}^2/\text{in.}$$

$$\mathbf{K} = \begin{bmatrix} 1000 & -500 & 0 \\ -500 & 750 & -250 \\ 0 & -250 & 250 \end{bmatrix} \text{ kips/in.}$$

With $\psi^T = [1 \quad 2 \quad 3]$, the generalized mass m^* and generalized stiffness k^* are obtained from

$$m^* = \psi^T\mathbf{M}\psi = 19 \text{ kip s}^2/\text{in.}$$

$$k^* = \psi^T\mathbf{K}\psi = 1250 \text{ kips/in.}$$

The generalized force is obtained from

$$p^* = -\psi^T\mathbf{M}\begin{bmatrix}1\\1\\1\end{bmatrix}\ddot{u}_g = -1170\sin(6\pi t) \text{ kips}$$

The equation of motion is given by

$$19\ddot{z} + 1250z = -1170\sin(6\pi t)$$

Problem 2.12

Taking moments about A shown in Figure S2.12

$$ml^2\ddot{\theta} + 2ka^2\theta + mgl\sin\theta = 0$$

For small vibrations $\sin\theta \approx \theta$, hence

$$ml^2\ddot{\theta} + \left(2ka^2 + mgl\right)\theta = 0$$

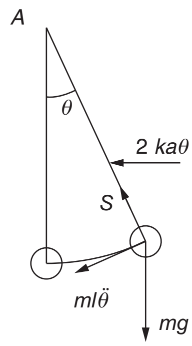

Figure S2.12

Chapter 3

Problem 3.1

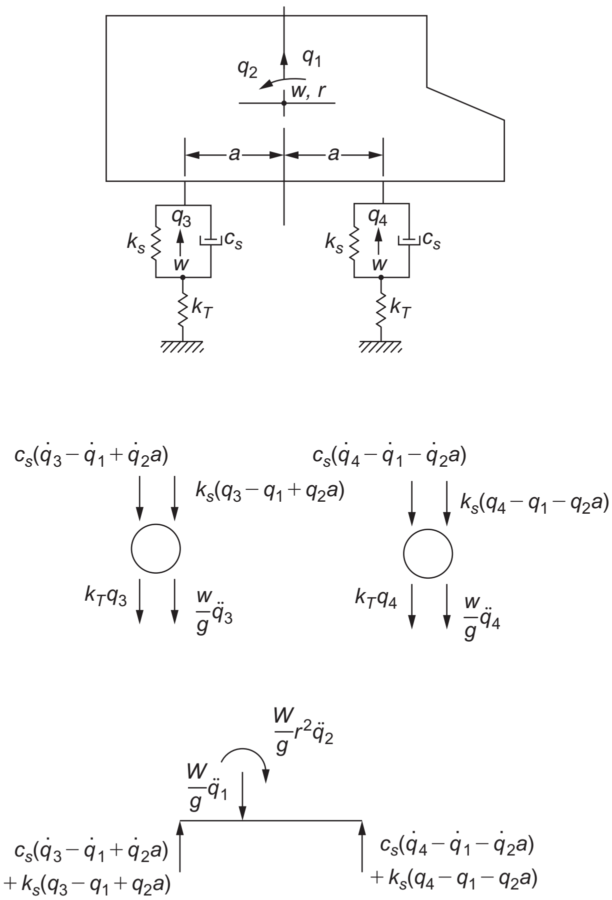

Figure S3.1

The degrees of freedom are identified in Figure S3.1. The free-body diagrams are also shown. For force balance on each of the front and rear axles

$$\frac{w}{g}\ddot{q}_4 + c_S\left(\dot{q}_4 - \dot{q}_1 - \dot{q}_2 a\right) + k_S\left(q_4 - q_1 - q_2 a\right) + k_T q_4 = 0$$

$$\frac{w}{g}\ddot{q}_3 + c_S\left(\dot{q}_3 - \dot{q}_1 + \dot{q}_2 a\right) + k_S\left(q_3 - q_1 + q_2 a\right) + k_T q_3 = 0$$

For equilibrium of vertical forces on the vehicle body

$$\frac{W}{g}\ddot{q}_1 + c_S\left(\dot{q}_3 + \dot{q}_4 - 2\dot{q}_1\right) + k_S\left(q_3 + q_4 - 2q_1\right) = 0$$

Taking moments about the mass center of the vehicle body

$$\frac{W}{g}r^2\ddot{q}_2 + c_S a\left(\dot{q}_3 - \dot{q}_4 + 2a\dot{q}_2\right) + k_S a\left(q_3 - q_4 + 2aq_2\right) = 0$$

Collectively, the four equations can be expressed in the matrix form

$$\mathbf{M}^*\ddot{\mathbf{u}} + \mathbf{C}^*\dot{\mathbf{u}} + \mathbf{K}^*\mathbf{u} = 0$$

$$\mathbf{u}^T = \left[\,q_1 \quad aq_2 \quad q_3 \quad q_4\,\right]$$

$$\mathbf{M}^* = \begin{bmatrix} \frac{W}{g} & & & \\ & \frac{W}{g}\frac{r^2}{a^2} & & \\ & & \frac{w}{g} & \\ & & & \frac{w}{g} \end{bmatrix}$$

$$\mathbf{C}^* = c_S\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}^* = k_S\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 1 + \frac{k_T}{k_S} & 0 \\ -1 & -1 & 0 & 1 + \frac{k_T}{k_S} \end{bmatrix}$$

Problem 3.2

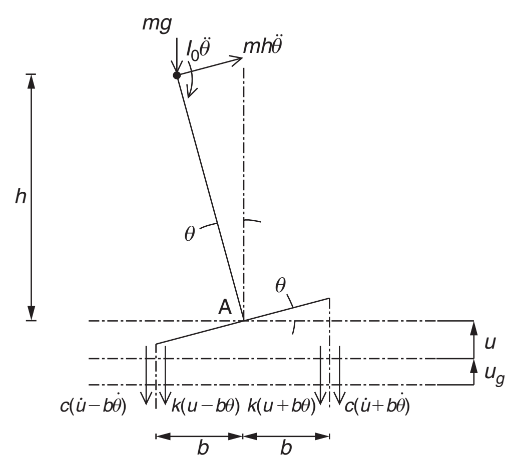

Figure S3.2

Coordinates u^t and θ measure the vertical motion and rotation of the base. The forces on the structure are identified in Figure S3.2. Assuming that u^t is measured from the position of equilibrium under gravity load, the condition of vertical force balance gives

$$m\ddot{u}^t + 2c\dot{u} + 2ku = 0 \tag{1}$$

or

$$m\ddot{u} + 2c\dot{u} + 2ku = -m\ddot{u}_g$$

Taking moments about A and assuming that θ is small

$$\left(I_0 + mh^2\right)\ddot{\theta} + 2cb^2\dot{\theta} + 2kb^2\theta - mgh\theta = 0 \tag{2}$$