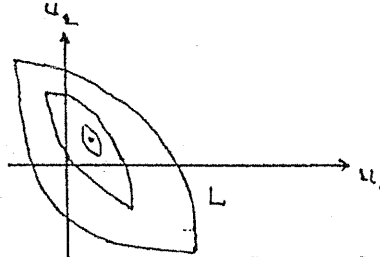


# Chapter 1

1.1-1

a).



$$u^* = -Q^{-1}S = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$L_{uu} = Q = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}.$$

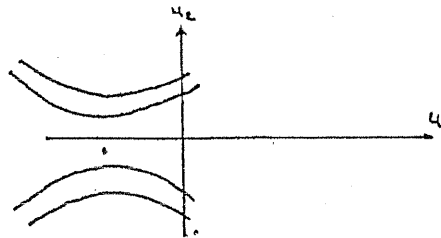
Since  $Q$  is negative definite,  $L_{uu} < 0$ . Therefore,  $u^*$  is local maximum.

$$L_u = \frac{1}{2} [u_1 \quad u_2] \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + [0 \quad 1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\frac{1}{2}u_1^2 + u_1u_2 - u_2^2 + u_2$$

Optimum value of  $L$ ,  $L^* = -1/2$ . The gradient

$$L_u = Qu + S = \begin{bmatrix} u_2 - u_1 \\ u_1 - 2u_2 + 1 \end{bmatrix}.$$

b).



$$u^* = -Q^{-1}S = -\frac{1}{3} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$L_{uu} = Q = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since  $|Q| \neq 0$ ,  $u^*$  is a saddle point.

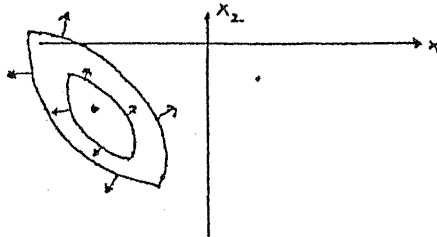
1.1-2

$$L(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1$$

$$\left. \begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 - x_2 + 3 = 0 \\ \frac{\partial L}{\partial x_2} &= -x_1 + 2x_2 = 0 \end{aligned} \right\} \Rightarrow x_1^* = -2, \quad x_2^* = -1.$$

$$L = -\frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [3 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$L_{uu} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0.$$



1.1-3

a).  $u = \begin{bmatrix} x \\ y \end{bmatrix}$ ,  $f_u = \begin{bmatrix} 2x \\ 4y^3 \end{bmatrix}$ ,  $f_u|_{u=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

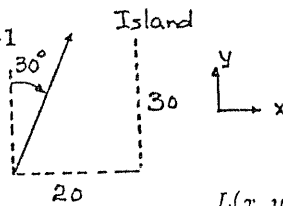
b).

$$f_{uu} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad f_{uu}|_{u=0} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

$|f_{uu}| = 0 \Rightarrow$  origin is a singular point.

c).  $f(x, y) = x^2 + y^4$ .  $f(0, 0) = 0$ ,  $x^2 > 0$  for all  $x \neq 0$ .  $y^4 > 0$  for all  $y \neq 0$ .  
 $f(x, y) = x^2 + y^4 > 0$  for all  $x \neq 0$  and  $y \neq 0 \Rightarrow (0, 0)$  is a minimum point.

1.2-1



$$L(x, y) = \frac{1}{2}(x - 20)^2 + \frac{1}{2}(y - 30)^2$$

$$f(x, y) = \sqrt{3}x - y = 0,$$

$$H(x, y) = L + \lambda^T f = \frac{1}{2}(x - 20)^2 + \frac{1}{2}(y - 30)^2 + \lambda(\sqrt{3}x - y),$$

$$\left. \begin{aligned} H_\lambda &= \sqrt{3}x - y = 0 \\ H_x &= x - 20 + \lambda\sqrt{3} = 0 \\ H_y &= y - 30 - \lambda = 0 \end{aligned} \right\} \Rightarrow \begin{aligned} \lambda &= 1.16 \\ x &= 17.99 \\ y &= 31.16 \end{aligned}$$

Closest point:  $(x, y) = (17.99, 31.16)$ , distance =  $\sqrt{2L^*} = 2.32$ .

Time =  $\frac{\sqrt{x^2 + y^2}}{10 \text{ mph}} \sim 36$  hours.

1.2-2

$$L = \min_{x,y}(d_1^2),$$

$$f = d_1^2 - d_2^2.$$

$$H = H(x, y, \lambda) = (x - x_1)^2 + (y - y_1)^2$$

$$+ \lambda [(x - x_1)^2 + (y - y_1)^2 - (x - x_2)^2 - (y - y_2)^2]$$

$$\frac{\partial H}{\partial x} = 2(x - x_1) + 2\lambda(x - x_1) - 2\lambda(x - x_2) = 0$$

$$\frac{\partial H}{\partial y} = 2(y - y_1) + 2\lambda(y - y_1) - 2\lambda(y - y_2) = 0$$

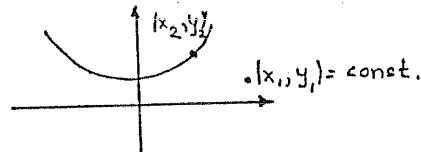
$$\frac{\partial H}{\partial \lambda} = (x - x_1)^2 + (y - y_1)^2 - (x - x_2)^2 - (y - y_2)^2 = 0$$

$$x = (1 + \lambda)x_1 - \lambda x_2$$

$$y = (1 + \lambda)y_1 - \lambda y_2$$

$$\lambda = -y_2, \quad x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

1.2-3



$$H = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}(y_1 - y_2)^2 + \lambda \left( \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1 \right)$$

$$H_{x_2} = -(x_1 - x_2) + \frac{2\lambda x_2}{a^2} = 0,$$

$$H_{y_2} = -(y_1 - y_2) + \frac{2\lambda y_2}{b^2} = 0,$$

$$H_{\lambda} = \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1 = 0.$$

$$\lambda = \frac{a^2(x_1 - x_2)}{2x_2} = -\frac{b^2(y_1 - y_2)}{2y_2}.$$

From the above equation we obtain

$$\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1,$$

$$a^2 x_1 y_2 + b^2 y_1 x_2 = (a^2 + b^2) x_2 y_2.$$

The above equations have two unknowns which can then be solved for.

1.2-4

a).

$$f(x, y) = x^2 + 3x - y - 6 = 0$$

$$L(x, y) = \frac{1}{2}(x-2)^2 + \frac{1}{2}(y-2)^2$$

$$H(x, y) = L + \lambda^T f = \frac{1}{2}(x-2)^2 + \frac{1}{2}(y-2)^2 + \lambda(x^2 + 3x - y - 6)$$

$$H_\lambda = x^2 + 3x - y - 6 = 0$$

$$H_x = x - 2 + \lambda(2x + 3) = 0$$

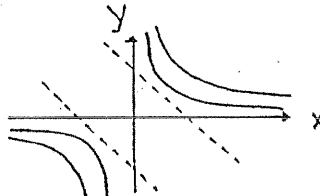
$$H_y = y - 2 - \lambda$$

$$y = \frac{3x + 8}{2x + 3}, \quad \lambda = \frac{2 - x}{2x + 3}, \quad 2x^3 + 9x^2 - 6x - 26 = 0,$$

b). By Newton's method ( $x_0 = 1.5$ , four iterations).  $x \sim 1.7086$ ,  $y \sim 2.0454$ ,  $\lambda \sim 0.0454$ , distance  $= \sqrt{2L^*} \sim 0.295$ .

1.2-5

a).



$$H(x, y) = xy + \lambda(2x + 2y - p)$$

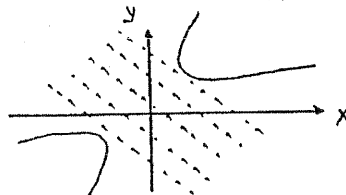
$$H_\lambda = 2x + 2y - p = 0$$

$$H_x = y + 2\lambda = 0$$

$$H_y = x + 2\lambda = 0$$

$$\lambda = -p/8, \quad x = y = p/4.$$

b).



$$H(x, y) = 2x + 2y + \lambda(xy - a^2)$$

$$H_\lambda = xy - a^2 = 0$$

$$H_x = 2 + \lambda y = 0$$

$$H_y = 2 + \lambda x = 0$$

$$\lambda = \frac{\sqrt{2}}{a}, \quad x = y = \sqrt{2}a.$$

1.2-6

$$\begin{aligned}
L &= \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \\
x &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u, \\
f &= x + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0, \quad B = \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \\
H &= \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u + \lambda^T \left\{ x + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}, \\
H_\lambda &= x + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0, \\
H_x &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \lambda = 0, \\
H_y &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} \lambda = 0, \\
u^* &= -(R + B^T Q B)^{-1} B^T Q C = \begin{bmatrix} -3 \\ 13/5 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 \\ -0.4 & 1 \end{bmatrix}, \\
x^* &= -(I - BK)C = \begin{bmatrix} 1.8 \\ 6 \end{bmatrix}, \quad \lambda^* = -Qx = \begin{bmatrix} -1.8 \\ -12 \end{bmatrix}, \\
L^* &= \frac{1}{2}C^T \lambda = \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix}.
\end{aligned}$$

1.2-7

a).

$$\begin{aligned}
K &= (R + B^T Q B)^{-1} B^T Q \\
U^* &= -KC \\
x^* &= -(I - BK)C \\
\lambda^* &= Q(I - BK)C \\
L^* &= \frac{1}{2}C^T Q[I - BK]C
\end{aligned}$$

b).

$$\begin{aligned}
S_0 &= Q - QB(B^T QB + R^{-1})B^T Q = Q(I - BK) \\
K &= (R + B^T QB)^{-1} B^T Q \\
0 &= B^T Q - (R + B^T QB)K
\end{aligned}$$

Furthermore,

$$\begin{aligned} B^T Q(I - BK) - RK &= 0 \\ -K^T B^T Q(I - BK) + K^T RK &= 0 \\ Q(I - BK) - K^T B^T Q(I - BK) + K^T RK &= Q(I - BK) \end{aligned}$$

where  $K^T B^T Q(I - BK) + K^T RK = 0$ . So,

$$\begin{aligned} S_0 &= Q(I - BK) = (I - K^T B^T)Q(I - BK) + K^T RK \\ &= (I - BK)^T Q(I - BK) + K^T RK \\ L^* &= \frac{1}{2} C^T S_0 C = \frac{1}{2} C^T (I - BK)^T \sqrt{Q}^T \sqrt{Q} (I - BK) C \\ &\quad + \frac{1}{2} C^T K^T \sqrt{R}^T \sqrt{R} K C \\ &= \frac{1}{2} \left[ \sqrt{Q} (I - BK) C \right]^T \left[ \sqrt{Q} (I - BK) C \right] + \frac{1}{2} \left( \sqrt{R} K C \right)^T \sqrt{R} K C \end{aligned}$$

c).

$$\begin{aligned} S_0 &= Q - QB(R + B^T QB)^{-1} B^T Q \\ &= (Q^{-1} + BR^{-1} B^T)^{-1} \end{aligned}$$

1.2-8

a).

$$\begin{aligned} L &= x^2 y^2 z^2, \quad f = x^2 + y^2 + z^2 - r^2 = 0 \\ H &= L + \lambda^T f = x^2 y^2 z^2 + \lambda^T (x^2 + y^2 + z^2 - r^2) \\ H_\lambda &= x^2 + y^2 + z^2 - r^2 = 0 \\ H_x &= 2xy^2 z^2 + 2x\lambda = 0 & (1) \\ H_y &= 2x^2 yz^2 + 2\lambda y = 0 & (2) \\ H_z &= 2x^2 y^2 z + 2z\lambda = 0 & (3) \end{aligned}$$

From (1)-(3), we have  $x^2 = y^2 = z^2$ . Now,

$$\begin{aligned} H_\lambda &= 3x^2 - r^2 = 0 \Rightarrow x^2 = \frac{r^2}{3} \\ L^* &= \left( \frac{r^2}{3} \right)^3 \end{aligned}$$

b).

$$H = x^2 + y^2 + z^2 + \lambda \left( x^2 y^2 z^2 - \left( \frac{r^2}{3} \right)^3 \right)$$

$$\begin{aligned}
H_\lambda &= x^2 y^2 z^2 - \left(\frac{r^2}{3}\right)^3 = 0 \\
H_x &= 2x + 2x\lambda y^2 z^2 = 0 \Rightarrow y^2 z^2 = -1/\lambda \\
H_y &= 2y + 2y\lambda x^2 z^2 = 0 \Rightarrow x^2 z^2 = -1/\lambda \\
H_z &= 2z + 2z\lambda x^2 y^2 = 0 \Rightarrow x^2 y^2 = -1/\lambda.
\end{aligned}$$

Therefore,

$$\begin{aligned}
x^2 &= y^2 = z^2 \Rightarrow x^2 = r^2/3 \\
L^* &= x^2 + y^2 + z^2 = r^2.
\end{aligned}$$

c).

$$\begin{aligned}
L &= b_1^2 b_2^2 b_3^2 \cdots b_n^2 \\
f &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2 - r^2 = 0 \\
H &= L + \lambda f \\
H_\lambda &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2 - r^2 = 0 \tag{4}
\end{aligned}$$

$$H_{b_1} = 2b_1 b_2^2 b_3^2 \cdots b_n^2 + 2b_1 \lambda = 0 \tag{5}$$

$\vdots$

$$H_{b_n} = 2b_1^2 b_2^2 b_3^2 \cdots b_n + 2b_n \lambda = 0 \tag{6}$$

From (4)-(6), we have

$$\begin{aligned}
b_1^2 &= b_2^2 = b_3^2 = \cdots = b_n^2 \\
H_\lambda &= nb_1^2 - r^2 = 0 \Rightarrow b_1^2 = r^2/n \\
L^* &= \left(\frac{r^2}{n}\right)^n = \left(\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}\right)^n \\
b_1^2 b_2^2 \cdots b_n^2 &\leq \left(\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}\right)^n
\end{aligned}$$

Let  $a_i = b_i^2$ . Hence,  $a_i > 0$  and  $\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$  is the maximum value of  $a_1 a_2 a_3 \cdots a_n$  i.e.,

$$\begin{aligned}
a_1 a_2 a_3 \cdots a_n &\leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \\
(a_1 a_2 a_3 \cdots a_n)^{\frac{1}{n}} &\leq \frac{(a_1 + a_2 + \cdots + a_n)}{n}
\end{aligned}$$

## 1.2-9

$$\begin{aligned} f_1(x, y, z) &= 3x + 2y + z - 1 = 0 \\ f_2(x, y, z) &= x + 2y - 3z - 4 = 0 \\ L(x, y, z) &= \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 \end{aligned}$$

Let

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$H = \frac{1}{2}w^T w + [\lambda_1 \quad \lambda_2] \begin{bmatrix} 3 & 2 & 1 & -1 \\ 1 & 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$H_{\lambda_1} = 3x + 2y + z - 1 = 0$$

$$H_{\lambda_2} = x + 2y - 3z + 4 = 0$$

$$H_x = y + 3\lambda_1 + \lambda_2 = 0 \quad (7)$$

$$H_y = y + 2\lambda_1 + 2\lambda_2 = 0 \quad (8)$$

$$H_z = z + \lambda_1 - 3\lambda_2 = 0 \quad (9)$$

From (7)-(9) we obtain

$$x = -3\lambda_1 - \lambda_2$$

$$y = -2\lambda_1 - 2\lambda_2$$

$$z = -\lambda_1 + 3\lambda_2$$

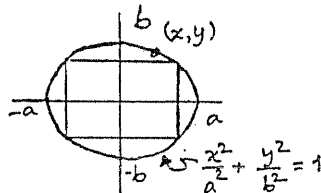
and

$$7\lambda_1 + 2\lambda_2 = -\frac{1}{2}$$

$$2\lambda_1 + 7\lambda_2 = -2$$

result in  $\lambda_1 = \frac{1}{90}$ ,  $\lambda_2 = -\frac{13}{95}$ . Hence,  $x = \frac{23}{90}$ ,  $y = \frac{5}{9}$ ,  $z = -\frac{79}{90}$ .

## 1.2-10



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a). Here we have,



$$\begin{aligned}
L(x, y) &= 4(x + y) \\
f(x, y) &= \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\
H(x, y) &= L + \lambda^T f = 4x + 4y + \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \\
H_x &= 4 + \frac{2\lambda}{a^2} x = 0 \tag{11}
\end{aligned}$$

$$H_y = 4 + \frac{2\lambda}{b^2} y = 0 \tag{12}$$

$$H_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{13}$$

Equations (11)-(13) result in

$$\begin{aligned}
\lambda &= -2\sqrt{a^2 + b^2} \\
x &= \frac{a^2}{\sqrt{a^2 + b^2}} \\
y &= \frac{b^2}{\sqrt{a^2 + b^2}}
\end{aligned}$$

$$\begin{aligned}
L''_{yy} &= H_{yy} - 2\frac{f_y}{f_x}H_{xy} + \frac{f_y^2}{f_x^2}H_{xx} \\
&= 2\frac{\lambda}{b^2} + \frac{2\lambda y^2 a^2}{x^2 b^4} = \frac{2\lambda}{b^2} \left[ 1 + \frac{a^2 y^2}{b^2 x^2} \right] < 0, \quad \text{iff } \lambda < 0
\end{aligned}$$

b).

$$\begin{aligned}
L(x, y) &= 4xy, \quad f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\
H(x, y) &= L + \lambda^T f = 4xy + \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \\
H_x &= 4y + \frac{2\lambda}{a^2} x = 0 \tag{14}
\end{aligned}$$

$$H_y = 4x + \frac{2\lambda}{b^2} y = 0 \tag{15}$$

$$H_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{16}$$

Equations (14)-(16) result in

$$\lambda = -2ab,$$

$$x = \frac{a}{\sqrt{2}},$$
$$y = \frac{b}{\sqrt{2}}$$

which gives the optimum area

$$L^* = \frac{ab}{2}.$$

This area is the maximum since

$$L''_{yy} = H_{yy} - 2\frac{f_y}{f_x}H_{xy} + \frac{f_y^2}{f_x^2}H_{xx}$$
$$= -4\frac{a}{b}\left[1 + \frac{ay}{bx}\right]^2 < 0, \quad (a > 0, b > 0).$$